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NOTE ON THE THEORY OF FUNCTIONS.

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1. While the Theory of Functions of a Complex Variable, together with all the results, expansions, etc. of that theory apply in general to the case when the variable is assumed to be real, care must be taken in this particularization of the variable in order to give the proper interpretation to the results, which at first sight may not be evident. This is one of the difficulties which presents itself to the student on passing from the real to the complex. He is too apt to feel that between the region of the real and that of the imaginary, there lies an uncertain ground in which the danger signals to be avoided are too frequently displayed. This field of apparent separation is too humble a one for cultivation by the master analysts to whose minds no difficulty presents itself, and who therefore deem the field a barren one. But few are endowed with the creative inspiration and can serve as pioneers in exploratory investigation, and the work of file-closing seems to me to be at the present stage of mathematical work not less important.

2. The foundation of Cauchy's theory of the complex function reposes on the theorem resulting from the integration, around a complete boundary, of a holomorphic function having within the boundary a single pole. In symbols this theorem is expressed by

$$\int_{(C)} \frac{f(z)}{z-a}\, dz = 2\pi i f(a)\,,$$

in which $f(z)$ is a function of $z = x + iy$, which is holomorphic throughout the area including the boundary determined by C, and a is a point of this area, i being $+\sqrt{-1}$. If we place $y - 0$, we assume the variable to be real. What then is the corresponding interpretation which must be given to Cauchy's theorem? It is the object of this note to attempt the answer to this question, and to interpret some of the simpler theorems which flow from it. In order to essay this, we approach the problem wholly from the domain of real quantity, remarking subsequently upon the correlation between the real and the complex.

3. Let $f(x)$ be a uniform, finite, and continuous function of the real variable x throughout the interval from $x = \alpha$ to $x = \beta$, inclusive. We call the integral

$$\int_{(\alpha\beta)} f(x)\, dx\,,$$

in which the integral is taken from α to β and then back to α again, the *complete integral* of the function throughout the interval $(\alpha\beta)$. This integral, when $f(x)$ is uniform, finite, and continuous throughout $(\alpha\beta)$, is evidently zero. Consider the complete integral

$$\int\limits_{(\alpha\beta)} \frac{f(x)}{x-a}\,dx\,,$$

in which $\alpha < a < \beta$. Let ε be a small *positive* quantity. We have

$$\begin{aligned}\int\limits_{(\alpha\beta)} \frac{f(x)}{x-a}\,dx = \int\limits_{\alpha}^{a-\epsilon} \frac{f(x)}{x-a}\,dx + \int\limits_{a-\epsilon}^{a+\epsilon} \frac{f(x)}{x-a}\,dx + \int\limits_{a+\epsilon}^{\beta} \frac{f(x)}{x-a}\,dx \\ + \int\limits_{\beta}^{a+\epsilon} \frac{f(x)}{x-a}\,dx + \int\limits_{a+\epsilon}^{a-\epsilon} \frac{f(x)}{x-a}\,dx + \int\limits_{a-\epsilon}^{\alpha} \frac{f(x)}{x-a}\,dx\,.\end{aligned}$$

The function $f(x)/(x-a)$ being uniform, finite, and continuous throughout the intervals $(\alpha,\ a-\varepsilon)$ and $(a+\varepsilon,\ \beta)$, the complete integral evidently becomes

$$\int\limits_{(\alpha\beta)} \frac{f(x)}{x-a}\,dx = \int\limits_{a-\epsilon}^{a+\epsilon} \frac{f(x)}{x-a}\,dx + \int\limits_{a+\epsilon}^{a-\epsilon} \frac{f(x)}{x-a}\,dx\,,$$

and therefore has no value unless it be in the immediate neighborhood of the pole a. The function $f(x)/(x-a)$ being infinite at a, the above integrals require special consideration in the interval $(a-\varepsilon,\ a+\varepsilon)$. We have

$$\begin{aligned}\int\limits_{a-\epsilon}^{a+\epsilon} \frac{f(x)}{x-a}\,dx &= \int\limits_{a-\epsilon}^{a+\epsilon} \frac{f(x)-f(a)}{x-a}\,dx + \int\limits_{a-\epsilon}^{a+\epsilon} \frac{f(a)}{x-a}\,dx \\ &= \int\limits_{a-\epsilon}^{a} \frac{f(x)-f(a)}{x-a}\,dx + \int\limits_{a}^{a+\epsilon} \frac{f(x)-f(a)}{x-a}\,dx + f(a)\int\limits_{a-\epsilon}^{a+\epsilon} \frac{dx}{x-a}\,.\end{aligned}$$

If the function $f(x)$ has a determinate finite derivative throughout the interval $(a-\varepsilon,\ a+\varepsilon)$, we have

$$f(x)-f(a) = (x-a)\,f'(u)\,,$$

where u is some value of x in this interval between x and a. The integral is therefore

$$\int\limits_{a-\epsilon}^{a+\epsilon} \frac{f(x)}{x-a}\,dx = \int\limits_{a-\epsilon}^{a} f'(u)\,dx + \int\limits_{a}^{a+\epsilon} f'(u)\,dx + f(a)\,.\log(-1)\,.$$

Now let ϵ diminish indefinitely, and we have in the limit

$$\int_{a-\epsilon}^{a+\epsilon} \frac{f(x)}{x-a}\,dx = f(a)\,.\log(-1)\,,$$

since the first two integrals on the right evidently vanish.

No real quantity can express the logarithm of an essentially negative real quantity. We therefore recognize in log (— 1) an unreal form. Is the result nugatory? Shall we, assuming no further knowledge of the nature of the symbol log (— 1), consider it to be an unreal indeterminate absurdity and discontinue the investigation, or conceive it to be a determinate analytical constant subject to the fundamental laws of algebra and examine the consequences which flow from its use as such? Choosing the latter course, let us represent log (— 1) by the symbol j. We then have, for the complete integral

$$\int_{(\alpha\beta)} \frac{f(x)}{x-a}\,dx = 2j\,f(a)\,.$$

The integral having no appreciable value except in the neighborhood of a, we may conventionally say, the value of the complete integral throughout the interval $(\alpha\beta)$ is its value at the pole a, and write

$$\int_{(\alpha\beta)} \frac{f(x)}{x-a}\,dx = \int_{(a)} \frac{f(x)}{x-a}\,dx = 2j\,f(a)\,.$$

4. Cauchy's Theory of Functions of a Complex Variable is based on the introduction into analysis of the imaginary unit or operator $i = +\sqrt{-1}$, which is taken as the unit of imaginary metrical length. This is but a conventional selection, arbitrarily made, of one particular value (the simplest) of the general symbol

$$(\pm 1)^{2n\pm 1}\sqrt{(-1)^{2n\pm 1}}\,.$$

We conventionally represent by i the analytical constant derived by making $n = 0$ and using only the upper sign. In this Theory, we have

$$\log(a + ib) = \tfrac{1}{2}\log(a^2 + b^2) + i\tan^{-1}(b/a)\,,$$

which for $b = 0$ and a essentially real and negative, we have

$$\log(-1) = i\tan^{-1}(-0) = i(2n \pm 1)\pi\,,$$

or, log (— 1) is an imaginary angle, which we may take as the unit of imaginary angles, by selecting that particular value of the general symbol

$$\log(-1) = (2n \pm 1)\pi\,(\pm 1)^{2n\pm 1}\sqrt{(-1)^{2n\pm 1}}\,,$$

under the same convention which selects the analytical constant unit i, by making $n = 0$ and using only the upper sign, which gives

$$\log(-1) = i\pi = j\,.$$

It is obvious that no integer value of n can alter the value of $\log(-1)$ so far as absolute position angle is concerned. There is in this sense no more indetermination in the general value of $\log(-1)$ than there is in the general value of i. Any imaginary angle expressed in unit angles is therefore

$$\theta \log(-1) = j\theta\,,$$

θ being a real number. If we choose to give geometrical interpretation to j as an operator, then, θ being the number which measures a real angle with respect to an initial line, the effect of the operator j on θ is to turn the plane of the real angle about its initial line through a right angle.

5. The theorems which flow from the integral at the end of § 3, of course follow in the same manner as do the generalizations of Cauchy's formula for a complex variable. For example, differentiating

$$f(a) = \frac{1}{2j}\int_{(\alpha\beta)} \frac{f(x)}{x-a}\,dx \tag{1}$$

with respect to a, gives

$$\frac{f^{n}(a)}{n!} = \frac{1}{2j}\int_{(\alpha\beta)} \frac{f(x)}{(x-a)^{n+1}}\,dx\,. \tag{2}$$

Again, let

$$\varDelta^{n|h} f(a) = f(a+nh) - C_{n,1} f(a+\overline{n-1}\,h) + \ldots + (-1)^{n} f(a)$$

be the nth progressive difference* of $f(x)$ at a, h being the scale of difference, and the points $(a + rh)$ $(r = 0, \ldots, n)$ points in $(\alpha\beta)$. Then in virtue of (1), we easily find

$$2j\,\varDelta^{n|h} f(a) = \int_{(\alpha\beta)} f(x)\,dx\,.\,\varDelta^{n|-h}\frac{(-1)^{n}}{(x-a)^{1|h}}\,;$$

whence

$$\frac{\varDelta^{n|h} f(a)}{n!\,h^{n}} = \frac{1}{2j}\int_{(\alpha\beta)} \frac{f(x)}{(x-a)^{n+1|-h}}\,dx\,, \tag{3}$$

wherein $(x-a)^{n+1|-h} = (x-a)(x-a-h)\ldots(x-a-nh)\,.$

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* See Note on the Theory of Functions of a Real Variable. Annals of Mathematics, Vol. VIII, pp. 66, 67.

In like manner, if the points $(a - rh)$ $(r = 0, \ldots, n)$ are in $(\alpha\beta)$, we have for the regressive difference-ratio at a

$$\frac{\Delta^{n|-h} f(a)}{n!\, h^n} = \frac{1}{2j} \int_{(\alpha\beta)} \frac{f(x)}{(x - a)^{n+1|+h}}\, dx\,. \tag{4}$$

If the nth derivative of $f(x)$ is finite and determinate, making $h = 0$ in (3) or (4) we derive (2).

6. Consider the complete integral

$$\int_{(\alpha\beta)} \frac{f(t)\, dt}{(t - a_0) \ldots (t - a_n)}\,,$$

wherein $a_0, \ldots, a_n$ are any points in $(\alpha\beta)$, and t the running point in the interval. We have by the ordinary rule for expanding rational fractions

$$\frac{1}{(t - a_0) \ldots (t - a_n)} = \sum_{0}^{n}{}_{r} \frac{1}{(t - a_r)\, \varphi'(a_r)}\,,$$

wherein

$$\varphi(t) = (t - a_0) \ldots (t - a_n)\,,$$

and $\varphi'(a_r)$ is the derivative of $\varphi(t)$ with respect to t and t is replaced by a_r.

Therefore by the result of (1), we have*

$$\begin{aligned}
\frac{1}{2j} \int_{(\alpha\beta)} \frac{f(t)\, dt}{(t - a_0) \ldots (t - a_n)} &= \sum_{0}^{n}{}_{r} \frac{f(a_r)}{\varphi'(a_r)} \\
&= \sum_{0}^{n}{}_{r} \frac{f(a_r)}{(a_r - a_0) \ldots (a_r - a_{r-1})(a_r - a_{r+1}) \ldots (a_r - a_n)} \\
&= \sum_{0}^{n}{}_{r} (-1)^r \frac{\zeta^{\frac{1}{2}}(a_0, \ldots, a_{r-1}, a_{r+1}, \ldots, a_n)}{\zeta^{\frac{1}{2}}(a_0, \ldots, a_n)} f(a_r) \\
&= \frac{\begin{vmatrix} f(a_0) & 1 & a_0 & \ldots & a_0^{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f(a_n) & 1 & a_n & \ldots & a_n^{n-1} \end{vmatrix}}{\zeta^{\frac{1}{2}}(a_0, \ldots, a_n)}\,.
\end{aligned}$$

Otherwise, and very much more briefly, we need only observe that the complete integral through $(\alpha\beta)$ is but the sum of the complete integrals at the poles, and write

$$\int_{(\alpha\beta)} \frac{f(t)\, dt}{\varphi(t)} = \int_{(a_0)} \frac{f(t)\, dt}{\varphi(t)} + \ldots + \int_{(a_n)} \frac{f(t)\, dt}{\varphi(t)} = \sum_{0}^{n}{}_{r} \frac{f(a_r)}{\varphi'(a_r)}\,.$$

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* Muir's Determinants, p. 165.

Consider the function

$$\frac{\begin{vmatrix} f(x) & 1 & x & \dots & x^n \\ f(a_1) & 1 & a_1 & \dots & a_1^n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f(a_n) & 1 & a_n & \dots & a_n^n \\ f(a_0) & 1 & a_0 & \dots & a_0^n \end{vmatrix}}{\zeta^{\frac{1}{2}}(a_0, \dots, a_n)},$$

which vanishes at the points $a_0, \dots, a_n$. Its nth derivative vanishes at the point u between the greatest and least of the values $a_0, \dots, a_n$, and we have

$$\frac{f^n(u)}{n!} = \frac{\begin{vmatrix} f(a_0) & 1 & a_0 & \dots & a_0^{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f(a_n) & 1 & a_n & \dots & a_n^{n-1} \end{vmatrix}}{\zeta^{\frac{1}{2}}(a_0, \dots, a_n)},$$

$$= \frac{1}{2j} \int_{(\alpha\beta)} \frac{f(t)\,dt}{(t - a_0) \dots (t - a_n)}.$$

If in this result we let $a_0 = a_1 = \dots = a_n = a$, we get the expression (2), since then u is also equal to a. We notice that the ratio of the determinants above takes the form 0 0 when the a's become a. If we consider this we have

$$\frac{1}{2j} \int_{(\alpha\beta)} \frac{f(t)\,dt}{(t - x)(t - a_0) \dots (t - a_n)} = \frac{f^{n+1}(u)}{(n+1)!}$$

$$= \frac{\begin{vmatrix} f(x) & 1 & x & \dots & x^n \\ f(a_0) & 1 & a_0 & \dots & a_0^n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f(a_n) & 1 & a_n & \dots & a_n^n \end{vmatrix}}{\zeta^{\frac{1}{2}}(x, a_0, \dots, a_n)}.$$

In order to remove the indetermination in this last expression caused by the a's becoming equal to a, we operate on the numerator and denominator with

$$\left[\frac{\partial}{\partial a_1}\right]^1_{a_1 = a} \dots \left[\frac{\partial}{\partial a_n}\right]^n_{a_n = a},$$

and after simplification and expansion, we have*

$$\frac{1}{2j}\int_{(\alpha\beta)}\frac{f(t)\,dt}{(t-x)(t-a)^{n+1}}=\frac{f^{n+1}(u)}{(n+1)!} \tag{5}$$

$$=\frac{1}{(x-a)^{n+1}}\left\{f(x)-f(a)-(x-a)f'(a)-\dots\right.$$

$$\left.-\frac{(x-a)^n}{n!}f^n(a)\right\},$$

wherein u lies between x and a.

Another neat way of getting this result is to consider (5) in the form

$$\int_{(\alpha\beta)}\frac{\dfrac{f(x)-f(t)}{x-t}}{(t-a)^{n+1}}\,dt=\frac{f^{n+1}(u)}{(n+1)!}.$$

The integral on the left is, by (2), equal to

$$\frac{2j}{n!}\left[\frac{d}{dt}\right]^n_{t=a}\left[\frac{f(x)-f(t)}{x-t}\right],$$

since the function

$$\frac{f(x)-f(t)}{x-t}$$

has no pole in $(\alpha\beta)$. But

$$\left[\frac{d}{dt}\right]^n_{t=a}\left[\frac{f(x)-f(t)}{x-t}\right]=\frac{n!}{(x-a)^{n+1}}\left\{f(x)-f(a)-(x-a)f'(a)-\dots\right.$$

$$\left.\frac{(x-a)^n}{n!}f^n(a)\right\}.$$

Whence at once

$$f(x)=f(a)+(x-a)f'(a)+\dots+\frac{(x-a)^n}{n!}f^n(a)+\frac{(x-a)^{n+1}}{(n+1)!}f^{n+1}(u).$$

Again, obviously, the integral in (5) can have no appreciable value except in the neighborhood of the points x and a. We may therefore write it

$$\int_{(\alpha\beta)}\frac{f(t)\,dt}{(t-x)(t-a)^{n+1}}=\int_{(x)}\frac{(t-a)^{-n-1}f(t)}{t-x}\,dt+\int_{(a)}\frac{(t-x)^{-1}f(t)}{(t-a)^{n+1}}\,dt.$$

But

$$\int_{(x)}\frac{(t-a)^{-n-1}f(t)}{t-x}\,dt=2j\left[\frac{f(t)}{(t-a)^{n+1}}\right]_{t=x}=2j\,\frac{f(x)}{(x-a)^{n+1}}.$$

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* See Note on Theory of Functions of a Real Variable, ANNALS OF MATHEMATICS, Vol. VIII, pp. 65, *et seq.*

Also,

$$\int\limits_{(a)} \frac{(t-x)^{-1} f(t)}{(t-a)^{n+1}}\,dt = \frac{2j}{n!}\left[\frac{\partial}{\partial t}\right]^n_{t=a}\left[\frac{f(t)}{t-x}\right],$$

and

$$\frac{(x-a)^{n+1}}{n!}\left[\frac{\partial}{\partial t}\right]^n_{t=a}\left[\frac{f(t)}{x-t}\right] = f(a) + (x-a)f'(a) + \ldots + \frac{(x-a)^n}{n!}f^n(a),$$

which is the same result as above.

7. If we apply the formula (1) directly to the expansion of functions in series, as is done in the Theory of Holomorphic Functions of a Complex Variable, we have

$$2j\, f(x) = \int\limits_{(\alpha\beta)} \frac{f(t)}{t-x}\,dt,$$

x being a point in $(\alpha\beta)$. We also have the identity

$$\frac{1}{t-x} \equiv \frac{1}{(t-a)\left[1 - \dfrac{x-a}{t-a}\right]}$$

$$\equiv \frac{1}{t-a} + \frac{x-a}{(t-a)^2} + \ldots + \frac{(x-a)^n}{(t-a)^{n+1}} + \frac{(x-a)^{n+1}}{(t-x)(t-a)^{n+1}}. \qquad (6)$$

Let a be also a point in $(\alpha\beta)$; then, by substitution and application of (2), we have

$$f(x) = f(a) + (x-a)f'(a) + \ldots + \frac{(x-a)^n}{n!}f^n(a)$$

$$+ \frac{(x-a)^{n+1}}{2j}\int\limits_{(\alpha\beta)} \frac{f(t)}{(t-x)(t-a)^{n+1}}\,dt. \qquad (7)$$

But we have proved this last integral to be

$$2j\,\frac{f^{n+1}(u)}{(n+1)!},$$

u lying between x and a.

We notice that if we substitute for $(t-x)^{-1}$ its identical value obtained from

$$-\frac{1}{t-x} = \frac{1}{x-a} + \frac{t-x}{(x-a)^2} + \ldots + \frac{(t-x)^n}{(x-a)^{n+1}} + \frac{(t-a)^{n+1}}{(x-a)^{n+1}(x-t)},$$

instead of getting a form equivalent to Laurent's expansion we are led to the identity

$$\int_{(\alpha\beta)} \frac{f(t)}{t-x}\,dt = -\frac{1}{(x-a)^{n+1}} \int_{(\alpha\beta)} \frac{(t-a)^{n+1} f(t)}{t-x}\,dt$$
$$= -\frac{2j}{(x-a)^{n+1}} \left[(t-a)^{n+1} f(t)\right]_{t=x}$$
$$= 2j\,f(x),$$

the integrals of the other terms having vanished because the functions under the $\int$ signs have no pole in $(\alpha\beta)$.

8. Taking for granted that the *complete* integral may be applied with confidence to real functions of a real variable, we go further and apply it to a more general case. Consider the identity*

$$\frac{t}{t-x} \equiv 1 + \frac{x}{t-a_1} + \ldots + \frac{x(x-a_1)\ldots(x-a_{n-1})}{(t-a_1)\ldots(t-a_n)}$$
$$+ \frac{x(x-a_1)\ldots(x-a_n)}{(t-x)(t-a_1)\ldots(t-a_n)}.$$

The series on the right is convergent when extended to infinity, and has for its limit $t/(t-x)$ when the quantities involved are such that

$$\overset{\infty}{\underset{1}{\Pi_n}} \frac{x+a_n}{t+a_n} = 0,$$

which is true when

$$(x-t) \overset{\infty}{\underset{1}{\Sigma_n}} \frac{a_n}{t+a_n}$$

is infinite and negative.

Let $t = t - a_0$ and $x = x - a_0$. The identity may then be written

$$\frac{1}{t-x} \equiv \frac{1}{t-a_0} + \overset{n}{\underset{1}{\Sigma_r}}\, \overset{r}{\underset{0}{\Pi_p}} \frac{x-a_{p-1}}{t-a_p} + \overset{n}{\underset{0}{\Pi_p}} \frac{x-a_p}{(t-x)(t-a_p)}. \tag{8}$$

Now let $x, a_0, \ldots, a_n$ be any points in the interval $(\alpha\beta)$. If we multiply both sides of this identity by $f(t)\,dt$ and integrate completely over the interval $(\alpha\beta)$, we have

$$2j\,f(x) = 2j\,f(a_0) + \int_{(\alpha\beta)} \overset{n}{\underset{1}{\Sigma_r}}\, \overset{n}{\underset{0}{\Pi_p}} \frac{x-a_{p-1}}{t-a_p} f(t)\,dt + \int_{(\alpha\beta)} \overset{n}{\underset{0}{\Pi_p}} \frac{x-a_p}{t-a_p} \frac{f(t)}{t-x}\,dt. \tag{9}$$

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* Derived from Euler's identity. See Chrystal's Algebra II. 393; also, ANNALS OF MATHEMATICS, Vol. VIII, No. 2. Note on Theory of Functions of a Complex Variable.

But we have shown that

$$\int_{(\alpha\beta)} \overset{r}{\underset{0}{\Pi}}_p \frac{x - a_{p-1}}{t - a_p} f(t)\, dt = 2j \overset{r}{\underset{0}{\Pi}}_p (x - a_{p-1}) \frac{f^r(u_r)}{r\,!}, \tag{10}$$

wherein u_r is some point in $(\alpha\beta)$ lying between the greatest and least of $a_0, \ldots, a_n$; also that

$$\int_{(\alpha\beta)} \overset{n}{\underset{0}{\Pi}}_p \frac{x - a_p}{t - a_p} \frac{f(t)}{t - x}\, dt = 2j \overset{n}{\underset{0}{\Pi}}_p (x - a_p) \frac{f^{n+1}(u)}{(n+1)\,!}, \tag{11}$$

wherein u lies between the greatest and least of $x, a_0, \ldots, a_n$. We may therefore write this equality thus:

$$f(x) = f(a_0) + (x - a_0) f'(u_1) + \ldots + \overset{n}{\underset{0}{\Pi}}_p (x - a_{n-1}) \frac{f^n(u_n)}{n\,!}$$
$$+ \overset{n}{\underset{0}{\Pi}}_p (x - a_p) \frac{f^{n+1}(u)}{(n+1)\,!}.$$

It follows from this that whenever $f(x)$ can be expanded in an infinite series of powers of $x - a$ in the interval $(\alpha\beta)$, it can be expanded in an infinite series of the rational integral functions $(x - a_0) \ldots (x - a_r)$ $(r = 0, 1, 2, \ldots)$. Of course Taylor's series is the particular case when $a_0, \ldots, a_n$ all become a. For then u_r $(r = 1, \ldots, n)$ also become a, while u lies between x and a. (9) is then an interpolation formula; and we may suppose all the a's equal after the mth, say equal to a. Then we get the expansion of $f(x)$ in terms of the function at $a_0, \ldots, a_m$, and a and the successive derivatives at a.

In particular, let $a_0 = a$ and $a_p = ph$ $(p = 1, \ldots, n)$, and write

$$(t - a)^{r|-h} = (t - a)(t - a - h) \ldots (t - a - \overline{r - 1}\, h).$$

Then

$$\int_{(\alpha\beta)} \overset{r}{\underset{0}{\Pi}}_p \frac{x - a_{p-1}}{t - a_p} f(t)\, dt = \int_{(\alpha\beta)} \frac{(x - a)^{r|-h}}{(t - a)^{r+1|-h}} f(t)\, dt$$
$$= 2j \frac{(x - a)^{r|-h}}{r\,!} \frac{\Delta^r f(a)}{h^r},$$

and we have

$$f(x) = f(a) + \frac{(x - a)^{1|-h}}{1\,!} \frac{\Delta f(a)}{h} + \ldots + \frac{(x - a)^{n|-h}}{n\,!} \frac{\Delta^n f(a)}{h^n}$$
$$+ \frac{(x - a)^{n+1|-h}}{(n-1)\,!} f^{n+1}(u), \tag{12}$$

wherein u lies between x, a, and $a + nh$. If as n increases, we let h decrease so that $a + nh$ always remains in $(\alpha\beta)$, then this series passes into Taylor's for $n = \infty$.

9. The interpretation of § 4, which makes $\log(-1) = j = i\pi$ in particular, makes the foregoing results particularizations of the corresponding theorems in the theory of complex functions. The shrinkage of the boundary of integration to a double line causes the variable to pass through the poles and the argument of the function under consideration during the process of integration. This fact destroys the usefulness of the complete integral as a test of the disappearance of the remainder in series, and throws us back again, as it should, upon the test of the nth derivative. The results which flow from the primal assumption are consistent with the results of the theory of functions of a real variable in every particular; they are consistent also with the results of the theory of functions of a complex variable under the particularization of the variable, and serve to illustrate the transition between the two, besides pointing out, in a manner which I have not seen done otherwise, the advantages of the latter as the more perfect system of analysis. For myself, I at present fail altogether to discern wherein the process is at variance with rigorous analysis of uniform functions.

NOVEMBER, 1893.